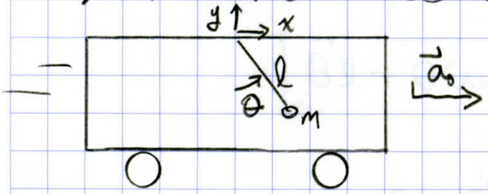
TM5 Pr 7.2

WORK OUT EXAMPLE 7.6 SHOWING ALL THE STEPS, INCLUDING THOSE LEADING TO 7.36 & 7.41. EXPLAIN WHY THE SIGN OF THE ACCELERATION CANNOT AFFECT ω. GIVE AN ARGUMENT WHY THE SIGNS OF a^2 AND g^2 ARE THE SAME.

EXAMPLE 7.6: FIND ω FOR A PENDULUM IN A TRAIN CAR.



SINCE THE ACCELERATION OF THE CAR ADDS TO THE PENDULUM'S HORIZONTAL MOTION, USE CARTESIAN COORDINATES.

$$\Rightarrow \quad x = (v_0 t + \tfrac{1}{2} a_0 t^2) + l \sin\theta$$

$$y = -l\cos\theta$$ ← $y=0$ AND $U=0$ AT TOP OF CAR

THE VELOCITIES ARE

$$\dot{x} = (v_0 + a_0 t) + l\dot{\theta}\cos\theta$$

$$\dot{y} = +l\dot{\theta}\sin\theta$$

AND THE ENERGIES CAN BE WRITTEN

$$U = -mgl\cos\theta$$

$$T = \tfrac{1}{2}m\left\{\left[v_0 + a_0 t + l\dot{\theta}\cos\theta\right]^2 + l^2\dot{\theta}^2\sin^2\theta\right\}$$

$$= \tfrac{1}{2}m\left\{v_0^2 + 2v_0 a_0 t + 2v_0 l\dot{\theta}\cos\theta + 2a_0 l\dot{\theta} t\cos\theta + a_0^2 t^2 + l^2\dot{\theta}^2\cos^2\theta + \right.$$
$$\left. + l^2\dot{\theta}^2\sin^2\theta\right\}$$ — $\sin^2\theta + \cos^2\theta = 1$

$$= \tfrac{1}{2}m\left\{v_0^2 + 2v_0 a_0 t + 2v_0 l\dot{\theta}\cos\theta + 2a_0 l\dot{\theta} t\cos\theta + a_0^2 t^2 + l^2\dot{\theta}^2\right\}$$

THE LAGRANGIAN IS

$$L = \tfrac{1}{2}m\left\{\left[v_0 + a_0 t + l\dot{\theta}\cos\theta\right]^2 + l^2\dot{\theta}^2\sin^2\theta\right\} + mgl\cos\theta$$

WHERE θ IS THE ONLY GENERALIZED COORDINATE

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TAKING DERIVATIVES

$$\frac{\partial L}{\partial \theta} = -m v_0 l \dot{\theta} \sin\theta - m a_0 l \dot{\theta} t \sin\theta - m g l \sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m v_0 l \cos\theta + m a_0 l t \cos\theta + m l^2 \dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml\left[-v_0 \dot{\theta} \sin\theta + a_0 \cos\theta - a_0 t \dot{\theta} \sin\theta + l\ddot{\theta}\right]$$

APPLY LAGRANGE'S EQUATION $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$

$$ml\Big\{\cancel{-v_0 \dot{\theta} \sin\theta} - \cancel{a_0 \dot{\theta} t \sin\theta} - g\sin\theta + $$
$$+ \cancel{v_0 \dot{\theta} \sin\theta} + \cancel{a_0 \dot{\theta} t \sin\theta} - a_0 \cos\theta - l\ddot{\theta}\Big\} = 0$$

$$\Rightarrow \quad g\sin\theta + a_0\cos\theta + l\ddot{\theta} = 0$$

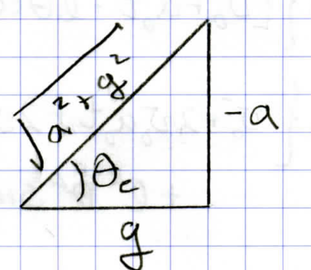
GIVING

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{a_0}{l}\cos\theta} \qquad (7.36)$$

THE EQUILIBRIUM ANGLE IS WHERE $\ddot{\theta} = 0$ AND $\theta = \theta_e$

$$\Rightarrow \quad g\sin\theta_e = -a_0\cos\theta_e$$

$$\Rightarrow \quad \tan\theta_e = \frac{-a}{g} \qquad (7.38)$$



SMALL OSCILLATIONS WILL BE ABOUT THE EQUILIBRIUM ANGLE θ_e, SO LET POSITION OF m BE DESCRIBED BY

$$\theta = \theta_e + \eta$$

WHERE η IS SMALL & THE SMALL ANGLE APPROXIMATIONS CAN BE USED

$\longrightarrow$

Substitute this into the angular acceleration

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta_e + \eta) - \frac{a_0}{l}\cos(\theta_e + \eta) = \ddot{\eta}$$

Expanding the angle sums

$$\ddot{\eta} = -\frac{g}{l}\left(\sin\theta_e\cos\eta + \cos\theta_e\sin\eta\right) - \frac{a_0}{g}\left(\cos\theta_e\cos\eta - \sin\theta_e\sin\eta\right)$$

Since η is small $\Rightarrow$ $\cos\eta \approx 1$ and $\sin\eta \approx \eta$

$$\ddot{\eta} = -\frac{g}{l}\left(\sin\theta_e + \eta\cos\theta_e\right) - \frac{a_0}{g}\left(\cos\theta_e - \eta\sin\theta_e\right)$$

$$= -\frac{1}{l}\left[\underbrace{\cancelto{0}{(g\sin\theta_e + a_0\cos\theta_e)}} + \eta\left(g\cos\theta_e - a_0\sin\theta_e\right)\right]$$

At equilibrium $g\sin\theta_e + a_0\cos\theta_e = 0$

$$\ddot{\eta} = -\frac{\eta}{l}\left(g\cos\theta_e - a_0\sin\theta_e\right) \qquad (7.40)$$

Recalling $\tan\theta_e = \frac{-a}{g}$ from previous page

$$\cos\theta_e = \frac{g}{\sqrt{a^2+g^2}} \quad \text{and} \quad \sin\theta_e = \frac{-a_0}{\sqrt{a^2+g^2}}$$

This can be written

$$\ddot{\eta} = -\frac{\eta}{l}\left(\frac{g^2}{\sqrt{a_0^2+g^2}} + \frac{a_0^2}{\sqrt{a_0^2+g^2}}\right) = -\frac{\eta}{l}\left(\frac{a_0^2+g^2}{\sqrt{a_0^2+g^2}}\right)$$

$$\Rightarrow \boxed{\ddot{\eta} + \frac{\sqrt{a_0^2+g^2}}{l}\,\eta = 0}$$

$\Rightarrow$ SHM with $\boxed{\omega_N = \frac{\sqrt{a_0^2+g^2}}{l}}$ QED (7.42)

- Since a_0 is squared, its direction (sign) does not affect ω_N.
- Signs of a_0^2 & g^2 are the same because the oscillations are about the hypotenuse of the triangle on the previous page.